Objectives:

- State the Fundamental Theorem of Calculus.
- Apply the Fundamental Theorem of Calculus to work with functions defined as integrals.

Example 1 If $g(x) = \int_1^x t^3\ dt$, find a formula for $g(x)$ that doesn't involve an integral and calculate $g'(x)$.

We can compute $g(x)$ with the Evaluation Theorem:

$$g(x) = \int_1^x t^3\ dt = \left.\frac{t^4}{4}\right|_1^x = \frac{x^4 - 1}{4}.$$

Now, we can take the derivative directly:

$$g'(x) = \frac{d}{dx}\left(\frac{1}{4}x^4 - \frac{1}{4}\right) = x^3$$

Weird! The derivative of $g(x) = \int_1^x t^3\ dt$ is eerily similar to the integrand, t^3.

The Fundamental Theorem of Calculus (Part I)

If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\ dt \qquad a \leq x \leq b$$

is an antiderivative of f, that is, $g'(x) = f(x)$ for $a < x < b$.

Example 2 Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}\ dt$.

Since $f(t) = \sqrt{1+t^2}$ is continuous for $x \geq 0$, we can use Part I of the Fundamental Theorem of Calculus:

$$g'(x) = \sqrt{1+x^2}.$$

The Fundamental Theorem of Calculus

Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\ dt$, then $g'(x) = f(x)$.
2. $\int_a^b f(x)\ dx = F(b) - F(a)$, where F is any antiderivative of f, that is $F' = f$.

For each of the following functions, find the derivative.

1. $g(x) = \int_1^x \frac{1}{t^3+1}\ dt$
   $g'(x) = \frac{1}{x^3+1}$
2. $r(y) = \int_2^y t^2 \sin(t)\ dt$
   $r'(y) = y^2 \sin(y)$
3. $F(x) = \int_x^\pi \sqrt{1+\sec(t)}\ dt$
   $F'(x) = -\sqrt{1+\sec x}$
4. $f(x) = \int_1^{x^4} \sec(t)\ dt$
   $f'(x) = \sec(x)(4x^3)$
5. $h(x) = \int_{2x}^{3x} \frac{u^2-1}{u^2+1}\ du$
   $h(x) = \int_{2x}^{c} \frac{u^2-1}{u^2+1}\ du + \int_c^{3x} \frac{u^2-1}{u^2+1}\ du$
   $= -\int_c^{2x} \frac{u^2-1}{u^2+1}\ du + \int_c^{3x} \frac{u^2-1}{u^2+1}\ du$
   $h'(x) = -\frac{(2x)^2-1}{(2x)^2}(2) + \frac{(3x)^2-1}{(3x)^2}(3)$